3.8 Inverse Functions:

Recall that 1–1 functions have inverses. A function is 1–1 if its graph satisfies the horizontal line test. The graph of the inverse function, $f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$.

Function	Domain	Inverse
$\sin x$	$-\pi/2 \le x \le \pi/2$	$\arcsin x$
$\tan x$	$-\pi/2 < x < \pi/2$	$\arctan x$
$\sec x$	$0 \le x \le \pi,\ x \ne \pi/2$	$\text{arcsec}\ x$
e^x	all x	$\ln x\ (x > 0)$
a^x	all x	$\log_a x\ (x > 0)$

Theorem: Assume $f(x)$ is one-to-one. If f is continuous then f^{-1} is continuous. If f is differentiable at x and $f'(x) \ne 0$ and then $f^{-1}(x)$ is differentiable at $y = f(x)$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ where } y = f(x).$$

Example: If $f(x) = x^2 + 4$, $x \ge 0$ then $f^{-1}(y) = (y-4)^{1/2}$ Since $f'(x) = 2x$, we have $x = f^{-1}(y) = (y-4)^{1/2}$

$$(f^{-1})'(y) = \frac{1}{2x} = \frac{1}{2(y-4)^{1/2}} = \frac{1}{2}(y-4)^{-1/2}$$

except when $y = 4$ which corresponds to $x = 0$.

Example Find the derivative of $\ln x$. By the theorem $\ln y$ is differentiable at every point $y = e^x$ so that $de^x/dx \ne 0$. Since $de^x/dx = e^x$ is never 0 $\ln y$ is differentiable for all y. Differentiate in the relation $e^{\ln x} = x$.

$$e^{\ln x}\frac{d}{dx}\ln x = \frac{d}{dx}x = 1$$

by the chain rule. Solve: Since $e^{\ln x} = x$

$$x\frac{d}{dx}\ln x = 1 \quad \text{so that} \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

We have therefore

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \quad u > 0$$

Example: Differentiate $y = 10^x$.

Solution: $y = 10^x = e^{x\ln 10}$. By the chain rule

$$y' = e^{x\ln 10}\ln 10 = \ln 1010^x$$

In general we have

$$\frac{d}{dx}a^u = (\ln a)a^u\frac{du}{dx}$$

Example: Differentiate $y = 2^{\sin x}$.
Solution: $y' = (\ln 2)2^{\sin x}\cos x = (\ln 2)(\cos x)2^{\sin x}$.
Example: Differentiate $y = e^{x^{1/2}}$.
Solution: By the chain rule,

$$y' = e^{x^{1/2}}\frac{1}{2}x^{-1/2} = \frac{e^{x^{1/2}}}{2x^{1/2}}$$

Example: Differentiate $y = \log_3 x$.
Solution: We use the identity $\log_3 x = \ln x / \ln 3$

$$y' = \frac{1}{(\ln 3)x}.$$

In general we have

$$\frac{d}{dx}\log_a u = \frac{1}{(\ln a)u}\frac{du}{dx}$$

Logarithmic Differentiation:
Example: Find the derivative of

$$g(x) = \frac{(x+2)^{1/3}(x^2+3)^{2/5}}{5x^3+x}$$

Solution: It is certainly possible to differentiate using the product, quotient and chain rules. However an alternative that is somewhat easier is logarthmic differentiation: Take logarithms

$$\begin{aligned}\ln g(x) &= \ln(x+2)^{1/3} + \ln(x^2+3)^{2/5} - \ln(5x^3+x)\\ &= \frac{1}{3}\ln(x+2) + \frac{2}{5}\ln(x^2+3) - \ln(5x^3+x)\end{aligned}$$

and then differentiate:

$$\frac{g'(x)}{g(x)} = \frac{1}{3}\frac{1}{x+2} + \frac{2}{5}\frac{2x}{x^2+3} - \frac{15x^2+1}{5x^3+x}$$

so that

$$g'(x) = \left[\frac{1}{3}\frac{1}{x+2} + \frac{2}{5}\frac{2x}{x^2+3} - \frac{15x^2+1}{5x^3+x}\right]\frac{(x+2)^{1/3}(x^2+3)^{2/5}}{5x^3+x}$$

Example: Differentiate $y = x^{\sqrt{x}}$. See text Example 4.
Solution: Differentiate $\ln y = x^{1/2}\ln x$ $y' = x^{-1/2}((1/2)\ln x + 1)y = x^{-1/2}((1/2)\ln x + 1)x^{\sqrt{x}}$